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On the critical behaviour of the anisotropic biquadratic spin-1 chain

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Abstract. A study is made in the anisotropic biquadratic spin-1 Hamiltonian introduced by Batchelor *et al*. This model can be represented as a sum of operators satisfying a Temperley–Lieb algebra. Its critical behaviour is explored by using the relations between mass-gap amplitudes and critical exponents predicted by conformal invariance. The model, like in the *XXZ* chain, is found to exhibit a line of continuously varying exponents. By comparing its eigenspectrum with that of the *XXZ* Hamiltonian, several exact values are predicted.

1. Introduction

The Yang–Baxter equations (YBE) play a central role in 2D exactly integrable models in statistical mechanics and conformal field theory. In the case of statistical mechanics these equations appear as conditions on the Boltzmann weights which ensure exact integrability [1]. In many cases these YBE solutions are consequences of the fact that the corresponding exactly integrable model can be expressed as a representation of special algebras like the Temperley–Lieb [3] and Hecke [4] algebras. Furthermore we expect that different representations of these algebras would correspond in principle to distinct exactly integrable models.

The Q-states Potts models and the spin-$\frac{1}{2}$ XXZ chain are the best known examples of different representations of the Temperley–Lieb algebras [3, 1]. These two models correspond to representations with dimensions Q and 4 respectively. It has also been shown [5] that a spin-1 model with a single quadratic term, namely the spin-1 biquadratic chain, is also a nine-dimensional representation of the Temperley–Lieb algebra, like the 9-state Potts models. From an exact equivalence between the eigenspectrum of these two last models it was conjectured that like the 9-state Potts model this spin-1 chain is also non-critical (massive).

More recently Batchelor *et al* [2] introduced the anisotropic biquadratic spin-1 model which is the deformed version of the above spin-1 model and is also a nine-dimensional representation of the Temperley–Lieb algebra. In this paper we calculate

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the critical properties of this anisotropic model. As in the XXZ model [6] we obtain, for a range of anisotropy values, critical behaviour with continuously varying exponents. Like most of the critical statistical mechanics models [7] this chain is also conformally invariant in its massless regime. The conformal anomaly and anomalous dimensions of the underlying field theory will be calculated. These calculations will be done by exploiting a set of important relations between these quantities and the eigenspectrum of the Hamiltonian with a finite number L of spins. These relations are consequences (see Cardy [7] for a review) of the conformal invariance of the infinite system at a critical point. The relevant relations, for our purposes, may be stated as follows. To each primary operator O_α, with anomalous dimension x_α and spin s_α, in the operator algebra of the massless infinite chain, there exists a set of states of the quantum Hamiltonian, in a periodic chain of L sites, whose energies and momenta are given by

$$E^{\alpha}_{j,j'}(L) = E_0(L) + 2\pi\zeta(x_\alpha + j + j')L^{-1} + o(L^{-1}) \tag{1.1a}$$

and

$$P^{\alpha}_{j,j'}(L) = 2\pi(s_\alpha + j - j')L^{-1} \tag{1.1b}$$

where j, $j' = 0, 1, 2, \ldots$, as $L \to \infty$. The ground-state energy of the finite chain is denoted by $E_0(L)$ and the (model-dependent) constant ζ is the sound velocity. In addition to these relations, conformal invariance also predicts [8] that, at criticality, the L-sites ground-state energy $E_0(L)$ in a periodic chain should behave as

$$\frac{E_0(L)}{L} = e_\infty - \frac{\pi c \zeta}{6} L^{-2} + o(L^{-2}) \tag{1.2}$$

as $L \to \infty$. Here c is the central charge of the conformal class governing the critical behaviour and e_∞ is the bulk limit ($L \to \infty$) of the ground-state energy per particle.

This paper is organized as follows. In section 2 we start by defining the model and discussing its main properties. Our main results are presented in sections 2.1–2.4. The ground-state energy of the infinite system and the sound velocity are calculated in section 2.1 and 2.2, and in sections 2.3 and 2.4 we calculate the conformal anomaly and dimensions in the critical region of the model. The paper closes with a discussion and a summary of our results in section 3.

2. The anisotropic biquadratic spin-1 antiferromagnetic chain

This model in an L-sites chain is defined by the Hamiltonian [2]

$$H_{bq}(\lambda_{bq}, L) = \sum_{k=1}^{L} H_{k,k+1}(\lambda_{bq}) \tag{2.1a}$$

where

$$\begin{aligned} H_{k,k+1}(\lambda_{bq}) = {} & -(\boldsymbol{S}_k \boldsymbol{S}_{k+1})^2 + \sinh^2(\lambda_{bq}) \left[S^z_k S^z_{k+1} - (S^z_k S^z_{k+1})^2\right] \\ & - \tfrac{1}{2}\sinh(\lambda_{bq}) \left[(S^x_k S^x_{k+1} + S^y_k S^y_{k+1})(S^z_{k+1} - S^z_k) + \mathrm{HC}\right] \\ & - 2\sinh^2\left(\tfrac{1}{2}\lambda_{bq}\right) \left[(S^x_k S^x_{k+1} + S^y_k S^y_{k+1}) S^z_k S^z_{k+1} + \mathrm{HC}\right] \\ & - \tfrac{1}{2}\sinh(2\lambda_{bq}) \left[S^z_k S^z_{k+1}(S^z_{k+1} - S^z_k)\right] \end{aligned} \tag{2.1b}$$

and $S_k = (S_k^x, S_k^y, S_k^z)$ are the spin-1 SU(2) matrices, λ_{bq} is a coupling constant and periodic boundary conditions are assumed. At $\lambda_{bq} = 0$ the model reduces to the isotropic spin-1 biquadratic chain [5, 9].

The main interesting feature of the above Hamiltonian [2, 10] is the fact that like the Q-states Potts quantum chain and the spin-$\frac{1}{2}$ *XXZ* chain, it can also be represented as a sum of operators

$$H_{bq}(\lambda_{bq}, L) = -\sum_{k=1}^{L}(U_k + 1) \tag{2.2a}$$

$$U_k \equiv -[H_{k,k+1}(\lambda_{bq}) + 1] \qquad k = 1, 2, \ldots, L \tag{2.2b}$$

satisfying the Temperley–Lieb algebra [3]

$$U_k^2 = \beta U_k \tag{2.3a}$$

$$U_k U_{k+1} U_k = U_k \tag{2.3b}$$

$$[U_k, U_{k'}] = 0 \qquad |k - k'| \geqslant 2 \tag{2.3c}$$

where

$$\beta = \beta_{bq} = \frac{\sinh(3\lambda_{bq})}{\sinh(\lambda_{bq})} = 1 + 2\cosh(2\lambda_{bq})\,. \tag{2.4}$$

In the case of the spin-$\frac{1}{2}$ *XXZ* chain

$$H_{xxz}(\gamma_{xxz}) = -\frac{1}{2}\sum_{k=1}^{L}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \Delta\sigma_k^z\sigma_{k+1}^z\right) \tag{2.5}$$

where now $(\sigma^x, \sigma^y, \sigma^z)$ are spin-$\frac{1}{2}$ Pauli matrices and $\Delta = -\cos(\gamma_{xxz})$ is a coupling constant, we can write (for periodic boundaries),

$$H_{xxz}(\gamma_{xxz}) = -\sum_{k=1}^{L}\left[U_k - \tfrac{1}{2}\cos(\gamma_{xxz})\right] \tag{2.6a}$$

where [3]

$$U_k = \frac{1}{2}\Big[(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y) + \cos(\gamma_{xxz})(1 - \sigma_k^z\sigma_{k+1}^z) + \frac{\mathrm{i}}{2}\sinh(\gamma_{xxz})(\sigma_{k+1}^z - \sigma_k^z)\Big] \tag{2.6b}$$

satisfy the Temperley–Lieb algebra given in (2.3) with

$$\beta = \beta_{xxz} = 2\cos(\gamma_{xxz})\,. \tag{2.6c}$$

In the case of the Q-states Potts chain the Hamiltonian is also expressed as a sum of operators obeying (2.3) but with [1, 3]

$$\beta_Q = \sqrt{Q}\,. \tag{2.7}$$

The operators U_k in these models, although satisfying the same algebra, have different dimensions. In the case of the spin-1 biquadratic model they are of dimension 9 and in the *XXZ* chain and Q-states Potts models their dimensions are 4 and 9, respectively. This fact was used by Barber and Batchelor [5] in order to show that the model (2.1) with $\lambda_{bq} = 0$ is massive because in this case $\beta_{bq} = \sqrt{9} = \beta_{Q=9}$ has the same value as in the 9-states Potts model (massive) and both model are expressed by Temperley–Lieb operators with same dimensions. The fact that the *XXZ* chain is critical for $-2 \leqslant \beta_{xxz} \leqslant 2$ as well as the Q-states Potts models with $\beta_Q \leqslant 2$ $(Q \leqslant 4)$ induce the conjecture that the anisotropic spin-1 biquadratic model given in (2.1) is also critical for $-2 \leqslant \beta_{bq} \leqslant 2$. In the remainder of this paper we verify this conjecture and the critical behaviour will be studied whenever it occurs.

2.1. *The ground-state energy in the bulk limit $L \to \infty$*

Using the relations (2.2) and (2.6) we can write for periodic chains

$$\frac{H_{bq}(\lambda_{bq}, L)}{L} \text{ `$=$' } \frac{H_{xxz}(\gamma_{xxz}, L)}{L} - \frac{\beta_{xxz}}{4} - 1 \tag{2.8a}$$

where the symbol '=' means that the left and right sides can be expressed in terms of different representions of the Temperley–Lieb algebra (2.3) but with the same parameter β:

$$\beta = \beta_{xxz} = 2\cos(\gamma_{xxz}) = \beta_{bq} = 1 + 2\cosh(2\lambda_{bq}). \tag{2.8b}$$

The number of eigenstates on the left of (2.8*a*) is 3^L while on the right side it is 2^L. Furthermore, we should not expect, for finite chains, common eigenvalues between these two models with boundary conditions of toroidal nature like the periodic one, because in this case the Temperley–Lieb operators $U_k; k = 1, 2, \ldots, L$ defined in (2.2) and (2.6) are not independent and should satisfy model-dependent constraints [11]. As usual we do, nevertheless, expect these spectral differences, induced by the boundaries, to vanish as $L \to \infty$ and consequently we expect that the ground-state energy per particle e_∞^{bq} and e_∞^{xxz} of both models, in the thermodynamic limit $(L \to \infty)$, are related by

$$e_\infty^{bq} = e_\infty^{xxz} - \tfrac{1}{4}\beta - 1 \tag{2.9}$$

with β given by (2.8*b*). The exactly-known bulk energy e_∞^{xxz} of the *XXZ* chain [6] gives us the conjectured asymptotic value which, for $-2 \leqslant \beta_{bq} = \beta_{xxz} \leqslant 2$, is

$$e_\infty^{bq}(\lambda_{bq}) = -1 - 2\sin^2(\gamma_{xxz}) \int_0^\infty \frac{\mathrm{d}x}{\cosh(\pi x)[\cosh(2\gamma_{xxz}x) - \cos(\gamma_{xxz})]} \tag{2.10}$$

where λ_{bq} and γ_{xxz} are related by (2.8*b*). In the case of non-toroidal boundary conditions both models in (2.8*a*) will have common eigenvalues even at finite lattices. Batchelor [18] has previously verified this fact.

The Hamiltonians (2.1) an (2.6) are U(1)-invariant due to their commutation with the total spin operator $\sum_{i=1}^{L} S_i^z$. Consequently their associated Hilbert spaces can be separated into block-disjoint sectors labelled by their total spin $n = \sum_{i=1}^{L} S_i^z = 0, \pm 1, \pm 2, \ldots$. As observed by Batchelor *et al* [2] the model (2.1) has a larger invariance, namely $U_q(SU_3)$. This invariance translates into a commutation of (2.1) with the operator $\sum_{i=1}^{L}(-1)^i[S_i^z]^2$. In this paper we do not explore this symmetry.

It is important to observe here that while the XXZ Hamiltonian is Hermitian for arbitrary values of β_{xxz} the related biquadratic model is Hermitian only for $\beta_{bq} \geqslant 3$, being non-Hermitian otherwise. Nevertheless our numerical calculations show that independently of β_{bq} the ground-state energy is real and occurs in the sector $n = 0$ due to the antiferromagnetic nature of (2.1). In order to check the validity of (2.10) we calculated numerically the eigenspectrum of (2.1) for several lattice sizes. Our results are shown in table 1. For $2 \leqslant L \leqslant 10$ the energies were calculated by a direct diagonalization while for $10 \leqslant L \leqslant 16$ the calculation was done by an extension of the Lanczos method for non-Hermitian matrices [12].

2.2. *The sound velocity*

A simple way to verify whether the Hamiltonian (2.1) is critical for $-2 \leqslant \beta \leqslant 2$, like the XXZ chain, is to analyse its eigenspectrum. In the case of massless behaviour, conformal invariance is also expected and the finite-size corrections of the eigenspectrum should be ruled by relations (1.1) and (1.2). In order to use these relations we should calculate the sound velocity. From (1.1) this constant, which is model-dependent, can be calculated from the difference between two consecutives energies e_1 and e_2 associated with the same conformal tower of a primary operator, i.e.

$$\Lambda_L = (e_2 - e_1)\frac{L^2}{2\pi} \to \zeta \,. \tag{2.11}$$

In table 2 we present the sequences Λ_L for some values of β_{bq}. In column A (column B) e_1 and e_2 are the lowest eigenenergies in the sector $n = 0$ (sector $n = 1$) having momentum 0 and $2\pi/L$, respectively. The extrapolated results, obtained by using the VBS extrapolants [13], are also shown in this table. Comparing these results with the exactly-known sound velocity of the XXZ chain [14, 11] we are induced to state the conjecture

$$\zeta = \frac{\pi \sin[\cos^{-1}(\beta_{bq}/2)]}{\cos^{-1}(\beta_{bq}/2)} \tag{2.12}$$

for the sound velocity of the Hamiltonian (2.1). In table 2 the conjectured results are also given. The agreement of the numerical estimatives with (2.12) is good for all values of $-2 \leqslant \beta_{bq} \leqslant 2$, except close to $\beta_{bq} = 2$ which is probably due to the appearance of logarithmic corrections like occur in the XXZ chain around $\gamma_{xxz} = 0$ ($\beta_{xxz} = 2$) [11].

2.3. *Conformal anomaly*

The results of the last section and our overall numerical analysis clearly indicate that the anisotropic spin-1 model is critical for $-2 \leqslant \beta_{bq} \leqslant 2$. From (1.2) the conformal anomaly c_{bq} of the conformal theory governing the critical behaviour of the model can be calculated form the $L \to \infty$ limit of the finite-size sequence

$$\hat{c}_{bq}(L) \equiv \frac{6L[e_\infty^{bq} L - E_0(\beta_{bq}, L)]}{\pi\zeta} \tag{2.13}$$

Table 1. Ground-state energy per particle of Hamiltonian (2.1). The conjectured results are given by (2.10).

L	$\beta_{bq} = -1.0$	$\beta_{bq} = 1.0$	$\beta_{bq} = 1.5$	$\beta_{bq} = 1.8$
4	−1.425 3905	−2.175 3905	−2.375 0000	−2.496 9641
6	−1.352 1676	−2.073 3098	−2.267 2385	−2.386 0545
8	−1.328 3474	−2.040 3561	−2.232 4713	−2.350 4515
10	−1.317 6272	−2.025 5681	−2.217 0502	−2.334 5367
12	−1.311 8839	−2.017 6573	−2.208 7607	−2.326 0473
14	−1.308 4478	−2.012 9288	−2.203 8118	−2.320 9841
16	−1.306 2285	−2.009 8787	−2.200 6204	−2.317 7215
Extrapolated	−1.299 036	−1.999 992	−2.190 307	−2.307 207
Exact	−1.299 0381	−2.000 0000	−2.190 3183	−2.307 2168

Table 2. Finite-size sequences of Λ_L for $L = 2, 16$ (see (2.11)). In the column A (column B) e_1 and e_2 are the lowest eigenenergies in the sector $n = 0$ (sector $n = 1$) of (2.1) having momentum 0 and $2\pi/L$, respectively. The conjectured values are given by (2.12).

	$\beta_{bq} = 0.1$		$\beta_{bq} = 1.0$		$\beta_{bq} = 1.5$	
L	A	B	A	B	A	B
4	2.045 25		2.356 48		2.546 47	
6	2.056 73	1.685 46	2.491 10	1.933 97	2.741 96	2.043 90
8	2.059 85	1.845 22	2.540 02	2.204 52	2.810 99	2.367 77
10	2.061 14	1.922 03	2.562 33	2.338 66	2.841 27	2.531 70
12	2.061 80	1.964 53	2.574 15	2.414 40	2.856 55	2.625 94
14	2.062 19	1.990 43	2.581 08	2.461 23	2.865 01	2.685 17
16	2.062 43		2.585 46		2.870 03	
Extrapotaled	2.063	2.064	2.598	2.579	2.879	2.876
Conjectured	2.063 19		2.598 07		2.875 14	

where $E_o(\beta_{bq}, L)$ is the ground-state energy of the periodic chain with L spins. Using the conjectured exact values of the bulk energy and sound velocity given by (2.10) and (2.12) we show these sequences in table 3 together with their estimated asymptotic values.

The results of table 3 are somehow surprising. From the existence of a critical behaviour in a range of anisotropy values we would expect $c_{bq} \geqslant 1$, but nevertheless constant for all the values of the anisotropy, as in the exactly integrable spin-S Heisenberg model [15, 16]. Furthermore we would guess the values $c_{bq} = 1$ or $c_{bq} = \frac{3}{2}$ for the conformal anomaly. The guess $c_{bq} = 1$ would be due to the fact that like the XXZ chain ($c = 1$) the Hamiltonian can be represented as a sum of Temperley–Lieb operators and the other guess $c = \frac{3}{2}$ is related with the fact that for chains with the same length, the dimensionality of the associated Hilbert space of (2.1) is the same as that of the exactly integrable spin-1 Heisenberg chain ($c = \frac{3}{2}$) [15, 16]. A similar result, with effective conformal anomaly changing with the coupling constant is also obtained in the spin-$\frac{1}{2}$ XXZ Hamiltonian given by (2.5), when a special

Table 3. Finite-size sequences $\bar{c}_{bq}(L)$ (see (2.13)) and extrapolated values for the conformal anomaly c_{bq}. The conjectured values are given by (2.20).

L	$\beta_{bq}=-1.0$	$\beta_{bq}=0.3$	$\beta_{bq}=0.5$	$\beta_{bq}=1.0$	$\beta_{bq}=1.5$	$\beta_{bq}=1.8$
4	2.97223	2.24707	2.18719	2.06288	1.96284	1.90973
6	2.81201	2.12116	2.06286	1.94005	1.83943	1.78530
8	2.75782	2.07953	2.02166	1.89862	1.79629	1.74056
10	2.73299	2.06065	2.00294	1.87952	1.82582	1.71852
12	2.71958	2.05050	1.99287	1.86911	1.76408	1.70569
14	2.71152	2.04442	1.98682	1.86279	1.80821	1.69738
Extrapotaled	2.68965	2.02769	1.97014	1.84462	1.78776	1.66260
Conjectured	2.68929	2.02768	1.97013	1.84464	1.73134	1.65749

type of toroidal boundary condition ($0 \leqslant \phi_{xxz} \leqslant 2\pi$)

$$\sigma^x_{L+1} \pm \mathrm{i}\sigma^y_{L+1} = \mathrm{e}^{\pm\phi_{xxz}}(\sigma^x_1 \pm \sigma^y_1) \qquad \sigma^z_{L+1} = \sigma^z_1 \tag{2.14}$$

is used [11]. The effect of these boundaries in the spectrum of the *XXZ* chain is to reduce the finite-size corrections producing the same corrections as a model with conformal anomaly

$$c_{xxz}(\phi_{xxz}) = 1 - \frac{3\phi^2_{xxz}}{2\pi(\pi - \gamma_{xxz})}. \tag{2.15}$$

Consequently it may also happen that the same role played by the boundary condition (2.14) in the *XXZ* chain is played by the periodic boundary condition in the Hamiltonian (2.1). Moreover it can also be shown [17] that the *XXZ* chain with Dzyaloshinsky–Moriya interactions and periodic ends is related to the *XXZ* chain (2.5) with toroidal boundary conditions of the type (2.14). Consequently if we measure the conformal anomaly for this last model we also obtain an effective conformal anomaly changing continuously with the coupling constant. In order to understand the results of table 3 we also calculated the eigenspectrum of our model (2.1) with the boundary condition

$$S^x_{L+1} \pm S^y_{L+1} = \mathrm{e}^{\pm\mathrm{i}\phi_{bq}}(S^x_1 \pm \mathrm{i}S^y_1) \qquad S^z_{L+1} = S^z_1 \,. \tag{2.16}$$

Comparing the eigenspectra of the *XXZ* chain (2.5) and the anisotropic spin-1 biquadratic model (2.1) with coupling constants related by (2.8*b*) and boundary conditions given by (2.14) and (2.16) we verify that the ground-state energy of both models, for finite chain are identical whenever

$$\cos(\tfrac{1}{2}\phi_{xxz}) = \cos(\phi_{bq}) + \tfrac{1}{2}\,. \tag{2.17}$$

Since the bulk energy e_{xxz} and e_{bq} should not depend on the boundary condition this result give us a strong argument in favour of the conjecture (2.10). It is also interesting to observe that the relation (2.17) between $\frac{1}{2}\phi_{xxz}$ and ϕ_{bq} is the same as that between γ_{xxz} and λ_{bq} in (2.8*b*). We cannot explain the reason why the ground-state energy of both model are equal, for finite chains, whenever conditions (2.8*b*) and (2.16) holds, but the reason is certainly related with the fact that the constraints produced by these toroidal boundary conditions in the representations (2.2) and (2.6) of the Temperley- Lieb algebra are the same, at least in a sector of the Hilbert space containing the ground state. The same type of exact relations for ground states was found previously between the Q-states Potts model and the XXZ chain [11].

Combining the results of (2.8), (2.15) and (2.17) we obtain the conjecture

$$c_{bq}(\beta,\phi_{bq}) = 1 - \frac{3}{2\pi[\pi - \cos^{-1}(\beta_{bq}/2)]}\tilde{\phi}_{bq}^2 \tag{2.18a}$$

where

$$\cos(\tfrac{1}{2}\tilde{\phi}_{bq}) = \cos(\phi_{bq}) + \tfrac{1}{2} \tag{2.18b}$$

for the effective conformal anomaly of the anisotropic spin-1 biquadratic model with boundary condition specified by the angle ϕ_{bq}. We also do expect that the above relations are valid for arbitrary values of ϕ_{bq} even if, due to relation (2.17), it would imply complex values for ϕ_{xxz}. In particular the periodic model ($\phi_{bq} = 0$) will be related to an XXZ chain with complex angle

$$\phi_{xxz} = \tilde{\phi}_{bq} = \Omega = \mathrm{i}2\ln\left(\frac{3+\sqrt{5}}{2}\right) \tag{2.19}$$

which give us, from (2.18), the conformal anomaly

$$c(\beta_{bq},0) = 1 + \frac{6}{\pi[\pi - \cos^{-1}(\beta_{bq}/2)]\ln^2[(3+\sqrt{5})/2]}\,. \tag{2.20}$$

These are the conjectured values presented in table 3, where we see a clear agreement with the estimated values.

It is interesting to observe from (2.18) that by choosing in the biquadratic model $\phi_{bq} = \pi/6$ we obtain $c_{bq}(\beta_{bq},\pi/3) = 1$ for arbitrary values of β_{bq}, like in the XXZ chain with periodic ends. This indicates that for this model the boundary condition (2.16) with $\phi_{bq} = \pi/3$ plays the same role as the periodic boundary condition in usual models. In order to elaborate more on this point let us split the boundary angles ϕ_{bq} into two regions: (i) $0 \leqslant \phi_{bq} < \pi/3$ where $\tilde{\phi}_{bq} = \mathrm{i}a,\ 2\ln(3+\sqrt{5}/2) \geqslant a > 0$ and (ii) $\phi_{bq} \geqslant \pi/3$ where $\tilde{\phi}_{bq}$ is real and grows from zero. While in region (i) the associated XXZ chain will have a boundary condition (2.14) with complex values for ϕ_{xxz}, in region (ii) this angle will be real. It is simple to see that the XXZ Hamiltonian (2.5) with boundary conditions specified by complex angles will lose its Hermiticity and complex eigenvalues are allowed in its eigenspectrum. On the other hand the Hamiltonian (2.1), although having a real trace, is non-Hermitian even for real values of the boundary angle ϕ_{bq}. However our spectral calculations show us that no matter what the value of β_{bq} in region (ii) we always obtain a real spectrum, in contrast with region (i) where complex eigenvalues appear as the lattice size increases. This information tells us that the lowest value of the ground-state energy, compatible with a real spectrum, is obtained with the boundary angle $\phi_{bq} = \pi/3$. For boundary angles $\phi_{bq} > \pi/3$ the eigenspectra are real but the energy increases while for $\phi_{bq} < \pi/3$ (including the periodic case), although the ground-state energy decreases, the eigenspectra are no longer completely real. Studying numerically and analytically the eigenspectra of the *XXZ* Hamiltonian (2.5) we verified that a similar behaviour also occurs when we use complex angles in (2.14). The ground-state energy is real and decreases but excited states with complex energy values occur. If we measure the effective conformal anomaly with these complex angles we obtain $c > 1$, like the Hamiltonian (2.1) with $\phi_{bq} < \pi/3$. These facts indicate that we should consider the spin-1 biquadratic model in its massless regime $-2 \leqslant \beta_{bq} \leqslant 2$, being governed by a $c = 1$ conformal theory like the *XXZ* chain. The boundary condition (2.16) with $\phi_{bq} > \pi/3$ and $\phi_{bq} < \pi/3$ produce dimensions related with conformal theories having $c < 1$ and $c > 1$, respectively

2.4. *Anomalous dimensions*

The results of last section indicate that we should expect anomalous dimensions given by a Coulomb gas picture like in the *XXZ* chain [11]. In the *XXZ* chain with periodic ends, the dimensions appearing in the sector $n = \sum_{i+1}^{L} \sigma_i^z = 0, \pm 1, \pm 2, \ldots,$ are given by [11]

$$x_{n,m} = \Delta_{n,m}^{+} + \Delta_{n,m}^{-} = n^2 x_p + \frac{m^2}{4x_p} \tag{2.21a}$$

where

$$\Delta_{m,n}^{\pm} = \frac{1}{2}\left(n\sqrt{x_p} \pm \frac{m}{2\sqrt{x_p}}\right)^2 \qquad x_p = \frac{\pi - \gamma_{xxz}}{2\pi}\,. \tag{2.21b}$$

Comparing the eigenspectra, for finite chains, of the biquadratic spin-1 model (2.1) and the *XXZ* chain (2.5), both models having periodic boundary conditions and anisotropy $\beta = \beta_{bq} = \beta_{xxz}$, we verify that although the ground-state energy of both models are different many real energies associated with excited states are exactly equal. Labelling by $n = \sum_{i=1}^{L} S_i^z (n = 0, \pm 1, \pm 2, \ldots, L)$ the eigensectors of the biquadratic model and by $n' = \sum_{i=1}^{L} \sigma_i^z (n' = 0, \pm 1, \pm 2, \ldots, \pm L/2)$ those of the *XXZ* chain we can verify the following exact correspondences. (i) If n is odd all the eigenenergies in the sector $n' = (|n| + 1)/2$ are present in sector n; (ii) if $n \neq 0$ and even all the eigenenergies in the sector $n' = n/2$ also occur in the sector n; (iii) if $n = 0$ or $n = 1$ all the eigenenergies in the sectors with $|n'| > |n|$ appear in the sector n, (iv) if $|n| > 1$ all the eigenenergies in the sector $|n'| \geqslant |n|$ appear in the sector n. These results imply, for example, that all the energies in the sector $n' \neq 0$ of the *XXZ* chain are also present in the eigenspectrum of the biquadratic model, when both models are subject to periodic boundary conditions.

Using these exact correspondences, which we believe are valid even for $L \to \infty$, and the relations (1.2), (2.2), (2.6) together with (2.21) we obtain in the sector $n = \sum_{i=1}^{L} S_i^z$ of the spin-1 biquadratic model the anomalous dimensions

$$d_{n,m} = x_{n',m} - x_{0,\Omega/2\pi} \tag{2.22a}$$

where

$$\begin{aligned} & n \text{ odd} \Rightarrow n' = \frac{|n|+1}{2} \\ & n \neq 0, \text{ even} \Rightarrow n' = \frac{n}{2} \\ & n = 0 \Rightarrow n' \neq 0 \\ & |n| = 1 \Rightarrow |n'| > |n| \\ & |n| > 1 \Rightarrow |n'| \geqslant |n| \end{aligned} \tag{2.22b}$$

and $x_{l,t}$ and Ω are given by (2.21) and (2.19), respectively. As a numerical test for (2.22) we show in table 4 our numerical estimate for the dimension $d_{1,0}$, obtained form the $L \to \infty$ of the sequence

$$\Lambda_{1,0}(L) = \frac{[E_{1,0}(\beta_{bq}, L) - E_0(\beta_{bq}, L)]L}{2\pi\zeta} \tag{2.23}$$

where $E_{1,0}$ is the lowest eigenenergy in the sector $n = 0$ and E_0 the ground-state energy of the L-sites chain. For the sake of comparison we also present our extrapolated results together with the conjectured values given by (2.22). Except around $\beta_{bq} = 2$, where logarithmic corrections are expected [11], the agreement is good. It is important to state here that the spin-1 biquadratic model should have other dimensions beyond these given by (2.22) because some of the higher energies in its spectrum (some of them complex) show no exact correspondence in the spectrum of the periodic *XXZ* spectrum. Before closing this chapter we mention that when the lattice size L is an odd number all the eigenvalues of the *XXZ* chain, including the ground-state sector $n' = 0$ are also present in the corresponding Hamiltonian (2.1) with $\beta = \beta_{bq} = \beta_{xxz}$. Consequently in this case the dimensions (2.22) should be replaced by those of the *XXZ* chain with an odd number of sites calculated in (11).

Table 4. Finite-size sequences $\Lambda_{1,0}(L)$ (see (2.23)) and extrapolated values for the dimension $d_{1,0}$. The conjectured values are given by (2.22*a*).

L	$\beta_{bq} = 0.1$	$\beta_{bq} = 0.5$	$\beta_{bq} = 1.5$
4	0.374182	0.392321	0.442843
6	0.359700	0.380086	0.440273
8	0.354898	0.376058	0.440686
10	0.352720	0.374236	0.441363
12	0.351548	0.373258	0.441966
14	0.350846	0.372671	0.442460
16	0.350391	0.372292	0.442861
Extrapotaled	0.34891	0.37106	0.44472
Conjectured	0.348914	0.371060	0.445919

3. Conclusion and summary

In this paper we have studied the anisotropic spin-1 biquadratric Hamiltonian (2.1) introduced by Batchelor *et al* [2]. This model, like the *XXZ* chain, has a U(1) invariance and can also be represented as a sum of Temperley–Lieb operators. Comparing the eigenspectrum of both models we conclude that like the *XXZ* chain this model has a massless phase when $-2 \leqslant \beta_{bq} = 1 + 2\cosh(2\lambda_{bq}) \leqslant 2$ with continuously varying exponents.

Exploiting the conformal invariance of the infinite system we calculate the conformal anomaly and dimensions of the underlying field theory governing the critical behaviour of the model. We show that this theory has a conformal central charge $c = 1$, but unlike the *XXZ* chain, the model with periodic boundary conditions shows an effective conformal anomaly which changes continuously with the coupling constant in the massless phase. This effect is the same as that appearing in the *XXZ* Hamiltonian (2.5) when toroidal boundary conditions of the type (2.14) are imposed with ϕ_{xxz} complex. The natural boundary condition, which gives $c = 1$ for $-2 \leqslant \beta_{bq} \leqslant 2$, is the toroidal boundary condition (2.16) with $\phi_{bq} = \pi/3$. When we use $\phi_{bq} > \pi/3$ we obtain $c < 1$ and real anomalous dimensions while $\phi_{bq} < \pi/3$, which includes the periodic case ($\phi_{bq} = 0$), give us $c > 1$ and complex dimensions will also occur, these complex dimensions being related with oscillatory behaviour of correlation functions. These results illustrate the danger we are exposed to when drawing conclusions just from the analysis of the eigenspectrum of a Hamiltonian with periodic boundary condition. The proper analysis should be done by looking for the most general boundary condition compatible with the symmetry of the model.

Several anomalous dimensions are derived by comparing the eigenspectrum of (2.1) with that of the *XXZ*. These dimensions, in the periodic case, are the same as those appearing in the *XXZ* chain (2.1) with a boundary condition (2.2) specified by a complex angle.

Finally we would like to mention that the exact relationship between eigenenergies of the *XXZ* chain and of the Hamiltonian (2.1) is a clear indication in favour of the exact integrability of the last model.

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